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Kalman Filtering for a Two-Layer, Two-Dimensional Shallow-Water Model

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1. Introduction

During the last decade interest in data assimilation for the atmosphere and oceans has increased. This interest can be attributed to the increase in the number and accurancy of data available. In meteorology many methods have been used for data assimilation; variational, statistical, and empirical (Bengtsson et al., 1981; Ghil, 1988). A particularly promising method, widely used in engineering, has been applied to numerical weather prediction by Ghil et al. (1981). This method, Kalman (1960), provides the best procedure for sequentially assimilating data into a dynamic model, under a set of reasonable assumptions (Jazwinski, 1970; Gelb 1974; Ghil, 1988).

In the study of Ghil et al. (1981), further refined by Cohn (1982), the model atmosphere was governed by a one-dimensional set of shallow-water equations. This model includes some interesting features of the large-scale atmospheric and oceanic system, in particular the two time-scale behavior of slow Rossby waves and fast Poincaré waves (Pedlosky, 1987; Ghil and Childress, 1987). In order for the sequential filter to provide an estimate of the slowly-evolving state of the system, the gain matrix was multiplied by a projection matrix onto the slow-wave subspace, giving rise to a modification of the standard Kalman filter. This modification is fairly general and does not depend on the model atmosphere being considered, but it might be too complicated to be applied to an operational numerical weather prediction model. A comparison between the operational procedures of optimal interpolation (OI) and the modified Kalman filter was made. Cohn (1982) showed the advantages of the latter and suggested how to improve the OI procedure so as to perform almost as well as the modified Kalman filter. A summary of this improved OI is given in Cohn et al. (1981).

A more realistic model was studied by Parrish and Cohn (1985). They considered a two-dimensional (2-D), single layer shallow-water model on an f-plane and showed that the Kalman filter can be implemented in two dimensions. The results for the forecast error correlations are markedly different from those obtained by OI, leading to a very different weighting of observations.

Parrish and Cohn (1985) used an ingenuous idea to make the implementation of the filter feasible for a system of flow equations. In what follows, we use their algorithm for an even more realistic model, i. e., we consider a two-layer, 2-D shallow-water model on a beta plane.

Two-layer shallow-water models have been widely used for different purposes. Takacs (1986), considered such a model on the sphere in order to compare its results with those of the Goddard Laboratory for Atmospheres (GLA) fourth order General Circulation Model, and better understand the latter. Sinton and Mechoso (1984), studied the nonlinear evolution of frontal waves with a similar model.

The main feature of this model which interests us is its ability to exhibit baroclinic instability. This property was studied by Pedlosky (1963), who gave instability conditions. The performance of the Kalman filter in the presence of numerical instability in the barotropic vorticity equation was studied by Miller (1986). The main question to be answered here is weather the filter allows sufficiently accurate tracking of the atmospheric state in the presence of rapid baroclinic developments.

2. Model Description

The atmosphere is treated as a system with two shallow layers of inviscid, constant-density immiscible fluids. In the 2-D case the equations for such a system are,

$$\frac{\partial}{\partial t}\vec{V}_k = -(\vec{V}_k \cdot \nabla)\vec{V}_k - f\,\hat{k}\times\vec{V}_k - \nabla\left[\alpha_k\,\phi_1 + \phi_2\right] \tag{2.1a}$$

$$\frac{\partial\phi_k}{\partial t} = -\nabla\cdot(\phi_k\,\vec{V}_k) \tag{2.1b}$$

for k=1,2, the upper and lower layers, respectively; $\vec{V}_k$ is the velocity vector, ϕ_k is the geopotential, f is the Coriolis parameter and the α's are constants, $\alpha_1=1$, $\alpha_2=\rho_1/\rho_2$, where ρ is density.

These equations can now be linearized about a basic state in which there is no wind in the y-direction, i. e., $V_k \equiv 0$, and the wind in the x-direction is given by $u_k = U_k(y)$, for each layer. On a beta plane at 45^{o}N, the 2-D equations (2.1) are given explicitly by,

$$\frac{\partial}{\partial t}w_k + U_k\frac{\partial}{\partial x}u_k + \frac{\partial}{\partial x}\left[\alpha_1\,\phi_1 + \phi_2\right] - \left(f - (U_k)_y\right)v_k = 0 \tag{2.2a}$$

$$\frac{\partial}{\partial t}v_k + U_k\frac{\partial}{\partial x}v_k + \frac{\partial}{\partial y}\left[\alpha_1\,\phi_1 + \phi_2\right] + f\,u_k = 0 \tag{2.2b}$$

$$\frac{\partial}{\partial t}\phi_k + U_k\frac{\partial}{\partial x}\phi_k + \Phi_k\left(\frac{\partial}{\partial x}u_k + \frac{\partial}{\partial y}v_k\right) + (\Phi_k)_y\,v_k = 0 \tag{2.2c}$$

where x and y point eastward and northward, respectively, while u_k, v_k and ϕ_k are the perturbation fields.

In order to write system (2.2) in a vector-matrix form one can introduce the notation $\vec{w} \triangleq (u_1, v_1, \phi_1, u_2, v_2, \phi_2)^T$ and then,

$$\frac{\partial}{\partial t}\vec{w} + \frac{\partial}{\partial x}(A\vec{w}) + \frac{\partial}{\partial y}(B\vec{w}) + C\vec{w} = 0 \qquad (2.3a)$$

where A, B and C are given by

$$A = \begin{vmatrix} U_1 & 0 & \alpha_1 & 0 & 0 & 1 \\ 0 & U_1 & 0 & 0 & 0 & 0 \\ \phi_1 & 0 & U_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & U_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & U_2 & 0 \\ 0 & 0 & 0 & \phi_2 & 0 & U_2 \end{vmatrix}, \qquad (2.3b)$$

$$B = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \Phi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \Phi_2 & 0 \end{vmatrix}, \qquad (2.3c)$$

and

$$C = \begin{vmatrix} 0 & -(f-U_{1y}) & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(f-U_{2y}) & 0 \\ 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} . \qquad (2.3d)$$

To solve system (2.3) we use the Richtmyer two-step version of the Lax-Wendroff scheme (see Richtmyer and Morton, 1967; Ghil et al., 1981). The first step to the finite-difference expression for $w^k_{ij} \equiv \vec{w}\big((i-1)\Delta x, (j-1)\Delta y, k\Delta t\big)$ (see Appendix C in Parrish and Cohn, 1985) at an intermediate location,

$$w^{k+1/2}_{i+1/2,j+1/2} = \mu_x \mu_y \, w^k_{i+1/2,j+1/2} - \frac{1}{2} L_{j+1/2} \, w^k_{i+1/2,j+1/2} , \qquad (2.4)$$

for $i=1,2,\ldots,I$ and $j=1,2,\ldots,J-1$, where

$$\Delta x = X/I \quad , \quad \Delta y = Y/(J-1) , \qquad (2.5a,b)$$

with X and Y being the east-west and north-south extents, respectively. The averaging operators μ_x and μ_y are given by

$$\mu_x w_{i,j} = \frac{1}{2} (w_{i+1/2,j} + w_{i-1/2,j}) , \qquad (2.6a)$$

$$\mu_y w_{i,j} = \frac{1}{2} (w_{i,j+1/2} + w_{i,j-1/2}) , \qquad (2.6b)$$

The differencing operators δ_x and δ_y are defined by

$$\delta_x \varpi_{i,j} = \varpi_{i+1/2,j} - \varpi_{i-1/2,j} \quad , \tag{2.7a}$$

$$\delta_y \varpi_{i,j} = \varpi_{i,j+1/2} - \varpi_{i,j-1/2} \quad . \tag{2.7b}$$

and the operator L_j is defined as

$$L_j = \lambda_x \mu_y \delta_x A_j + \lambda_y \mu_x \delta_y B_j + \Delta t \; \mu_x \mu_y C_j \quad , \tag{2.8}$$

with $\lambda_x = \Delta t / \Delta x$ and $\lambda_y = \Delta t / \Delta y$.

The second step uses the intermediate values to compute the state variables at the next time level

$$\varpi_{i,j}^{k+1} = \varpi_{i,j}^{k} - L_j \; \varpi_{i,j}^{k} \quad , \tag{2.9}$$

for $i=1,2,\ldots,I$ and $j=2,3,\ldots,J-1$. Based on (2.9) it is possible to construct the dynamics matrix Ψ, which is needed in the Kalman filter scheme. The boundary conditions can be included in the formulation of Ψ. It is important to notice that, in order to obtain baroclinic instability the free upper surface and the interface between the two layers of fluid must have nonzero mean slopes.

3. The Kalman Filter

Very good and clear presentations of the Kalman filtering procedure can be found in many places: Jazwinski (1970), gives a good mathematical description while Gelb (1974) or Brown (1983), are more application-oriented. The Kalman filter as applied to numerical weather prediction is presented by Ghil et al. (1981). In this section we recapitulate briefly the essentials.

Let us consider a discrete linear system, given in vector notation by

$$\vec{w}^t(k) = \Psi(k-1)\vec{w}^t(k-1) + \vec{b}^t(k-1) \tag{3.1}$$

for the discrete times $k = 1,2,...,$ where $\vec{w}^t(k)$ is the n-vector representing the true state of the system, Ψ is the n×n dynamics matrix and $\vec{b}(k)$ is a random n-vector which is white in time and unbiased with covariance $Q(k)$:

$$E[\ \vec{b}^t(k)] = 0\ ,\quad E[\ \vec{b}^t(k)(\vec{b}^t(l))^T] = \delta_{kl}Q(k). \tag{3.2a,b}$$

The symbol E denotes the expectation, the superscript T denotes the transpose and δ_{kl} is the Kronecker delta.

Discrete linear observations are described by

$$\vec{w}^o(k) = H(k)\vec{w}^t(k) + \vec{b}^o(k)\ ,\quad k = 1,2,... \tag{3.3}$$

where $\vec{w}^o(k)$ is a p-vector of observations, and $H(k)$ is a p×n matrix accounting for interpolations and for any conversion between observed variables and state variables; $\vec{b}^o(k)$ is a random p-vector, unbaised with covariance matrix $R(k)$, describing observation error, and assumed to be uncorrelated with the model error $\vec{b}^t(k)$:

$$E[\ \vec{b}^o(k)] = 0\ , \quad E[\ \vec{b}^o(k)(\vec{b}^o(l))^T] = \delta_{kl} R(k)\ , \tag{3.4a,b}$$

$$E[\ \vec{b}^o(k)(\vec{b}^t(l))^T] = 0\ . \tag{3.4c}$$

The Kalman filter for the system described above is the following set of equations:

$$\vec{w}^f(k) = \Psi(k-1)\vec{w}^a(k-1)\ , \tag{3.5a}$$

$$P^f(k) = \Psi(k-1)P^a(k-1)\Psi^T(k-1) + Q(k-1)\ , \tag{3.5b}$$

$$K(k) = P^f(k)H^T(k)[\ H(k)P^f(k)H^T(k) + R(k)]^{-1}\ , \tag{3.5c}$$

$$P^a(k) = [\ I - K(k)H(k)]\ P^f(k)\ , \tag{3.5d}$$

$$\vec{w}^a(k) = \vec{w}^f(k) + K(k)[\ \vec{w}^o(k) - H(k)\vec{w}^f(k)]\ , \tag{3.5e}$$

for $k = 1,2,3,\ldots$. The superscripts "f" and "a" stand for forecast and analyzed quantities, that is, quantities that are due simply to the evolution of the physical system itself and those dependent on the observations, repectively. K introduced above is the Kalman gain matrix, which refers to the weight with which observations and forecast are combined to give the final result, and P is the error covariance matrix,

$$P^{f,a}(k) = E\Big[\ [\vec{w}^{f,a}(k)-\vec{w}^t(k)][\vec{w}^{f,a}(k)-\vec{w}^t(k)]^T\ \Big]\ . \tag{3.6}$$

for the forecast and analyzed fields, repectively.

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References

Bengtsson, L., M. Ghil and E. Källén (eds.), 1981: **Dynamic Meteorology: Data Assimilation Methods**. Springer-Verlag, New York, 330 pp.

Brown, R. G., 1983: **Introduction to Random Signal Analysis and Kalman Filtering**, John Wiley, New York, 347 pp.

Cohn, S. E., 1982: **Methods of Sequential Estimation for Determining Initial Data in Numerical Weather Prediction**. Ph.D. thesis, Courant Institute of Mathematical Sciences, NYU, New York, NY 10012, 183 pp.

Cohn, S. E., M. Ghil and E. Isaacson, 1981: **Optimal Interpolation and the Kalman Filter**. Fifth Conference on Numerical Weather Prediction, preprint volume, Amer. Meteor. Soc., Boston, pp. 36-42.

Gelb, A., 1974: **Applied Optimal Estimation**. Cambridge, MA, MIT Press, 374 pp.

Ghil, M., S. Cohn, J. Tavantzis, K. Buke and E. Isaacson, 1981: **Applications of estimation theory to numerical weather prediction.** Dynamic Meteorology: Data Assimilation Methods, L. Bengtsson, M. Ghil and E. Kallen (eds.), Springer-Verlag, New York, pp. 139-224.

Ghil, M., and S. Childress, 1987: **Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics**. Springer Verlag, New York, 485 pp.

Ghil, M., 1988: **Meteorological Data Assimilation for Oceanographers. Part I: Description and Theoretical Framework**. Dyn. Atmos. Oceans, submitted.

Jazwinski, A. H., 1970: **Stochastic Processes and Filtering Theory**. Academic Press, New York, 376 pp.

Kalman, R. E., 1960: **A New Approach to Linear Filtering and Prediction Problems**. Trans. ASME, Ser. D, J. Basic Eng., 82, 35-45.

Parrish, D. F. and S. E. Cohn, 1985: **A Kalman Filter for a Two-Dimensional Shallow-Water Model: Formulation and Preliminary Experiments**. Office Note 304, National Meteorological Center, Washington DC 20233, 64 pp.

Miller, R. M., 1986: **Toward the Application of the Kalman Filter to Regional Open Ocean Modelling**. J. Phys. Oceanogr., 16, 72-86.

Pedlosky, J., 1963: **Baroclinic Instability in Two-Layer Systems**. Tellus, 15, 20-25.

Pedlosky, J., 1987: **Geophysical Fluid Dynamics**. Springer Verlag, New York, 624 pp.

Richtmayer, R. D. and K. W. Morton, 1967: **Difference Methods for Initial-Value Problems, (2nd ed.)**. Wiley-Interscience, New York, 405pp.

Sinton, D. M. and C. R. Mechoso, 1984: **Nonlinear Evolution of Frontal Waves**. J. Atmos. Sci., 14, No. 24, pp. 3501-3517.

Takacs, L. L., 1986: **Documentation of the Goddard Laboratory for Atmospheres Fourth-Order Two-Layer Shallow Water Model.** Goddard Modelling and Simulation Branch, Goddard Space Flight Center, Greenbelt, Maryland 20771, 79 pp.